5 Toroidal Models in Tonal Theory and Pitch-Class Analysis

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Abstract

The doubly circular relations of the major and minor keys based on all twelve pitch-classes can be depicted in toroidal models. We demonstrate a convergence of derivations from the different bases of conventional harmonic theory and recent experiments in music psychology. We present a formalization of the music-theoretical derivation from Gottfried Weber's 1817 chart of tone-centers by using a topographic ordering map. We find the results to be consistent with Krumhansl and Kessler's 1982 visualization of perceptual ratings.

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5.1 Spatial Models of Key Relationships

Space is a natural medium for imagining relationships between percepts. Similarity of percepts can be identified with spatial proximity. Spatial distances can be represented mathematically by a similarity measure. Some integrated percepts can be decomposed into principal perceptual attributes that correspond to axes or their spatial projections. Perceptual models of emotions, timbre, pitch, and keys also use spatial representations to portray key relations. For a circular view of pitch-class, however, Euclidean space is not appropriate. Manifolds such as helices[1] and toroids are more suitable. Here we consider spatial representations of key relationships in tonal music in two contexts—that of music theorists and that of psychologists. We show the suitability of computer modeling procedures to express these relations spatially.

The geometrical models of tonal relations that emerge from such study seem to be influenced by the epistemological assumptions made. We have examined in detail three pathways to such models. They issue from analogue analyses in music literature, speculations of music theorists, and experiments by music psychologists. We have sought to identify equivalences among these approaches through the use of visualization and clustering algorithms such as principal components analysis, multidimensional scaling, independent component analysis, self-organizing feature maps, and correspondence analysis (Purwins 2005).

5.1.1 Heinichen's Circles

From the early eighteenth century until the appearance of Wagner's *Tristan* (1857), models of key relationships were largely dominated by tonic-dominant progressions and major-minor dualities. The most familiar of early geometrical explanations of key relationships (Figure 5.1) is given in Johann David Heinichen's *General-bass in der Composition* of 1728.[2] Heinichen's interest was prompted by the gradual adoption of equal-tempered tuning, which standardized the tuning of thirds, thus making the difference between major and minor modes more obvious than it had been when temperaments were more variable. Graceful modulation became a predominant interest in the nineteenth century, when a common system of tuning was taken for granted.

Figure 5.1. *J. D. Heinichen's chart of key relationships (1728: 837). Note that in contrast to later simplifications showing only major keys (in 12 positions), Heinichen interleaves relative minor keys (24 positions), which can be parsed into major- (here ma) minor (mi) pairs (e.g., 1 [C Major], 2 [A Minor], etc.).*

In Heinichen's Circle, 24 slots are filled alternately by major and minor keys. The C-G-D… Circle (odd numbers, major keys) is interleaved with the a-e-b… Circle (even numbers, minor keys). Note that the only tonal elements of this arrangement are minor thirds and perfect fifths. The many variants of Heinichen's Circle included those of David Kellner (*Treulicher Unterricht im General-Bass* [1732]) and Johann Mattheson (*Kleine General-Bass-Schule* [1735]). Kellner suggested decomposing the two nested circles into a single circle.

5.1.2 Weber's Grids

In search of a systematic explanation of tonal relations, the German composer, theorist, and inventor Gottfried Weber, in his *Versuch einer geordneten Theorie der Tonsetzkunst zum Selbstunterricht* (1817), dealt first with chord types. He distinguished three kinds of triads (major, minor, diminished) and four kinds of seventh chords [tetrads]. A long succession of charts describes increasingly more extensive ideas of tonal relations. In Figure 5.2, for example, we see twenty-four tones arrayed around C such that the vertical axis descends by fifths, while the horizontal axis traverses parallel major-minor key pairs. With regard to functional harmony, Weber emphasizes common tones between scales. That is, the dominant, subdominant, and relative major (minor) scales differ from that of the tonic by only one tone.[3] He considers the principal scale degrees to be I, IV, and V.

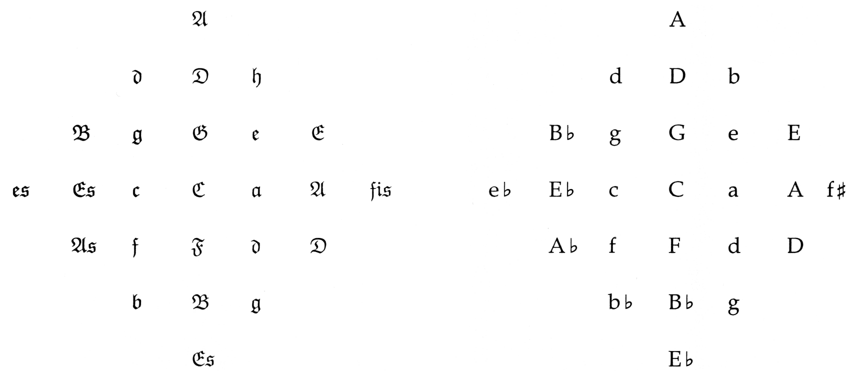

Figure 5.2. *Left: Facsimile of Gottfried Weber's schematic diagram of major-minor key relations (1817). D Major (𝔇), for example, is the parallel major of D Minor (𝔡) and the relative major of B Minor ([𝔥] h in German terminology). Right: transliteration of the Fraktur (German script) in Weber's diagram.*

In Weber's more extensive charts of tonal relations (Figure 5.3), we find multiple occurrences of the same (or enharmonically equivalent) tones among the 104 items. Figure 5.3 shows only the first panel of a three-page chart relating all major and minor keys. Multiple instances of several individual keys occur in the full chart.

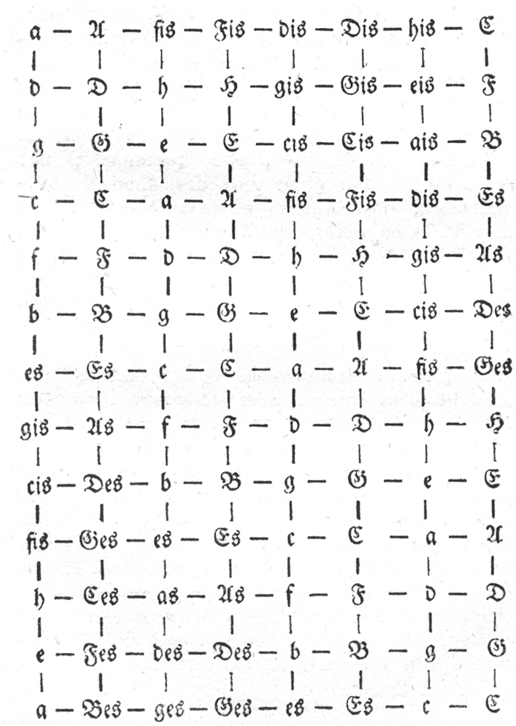

Figure 5.3. *By linking relative and parallel relations (with thirds on the horizontal plane and fifths on the vertical), Weber creates this schematic diagram of all major and minor keys.*

5.1.3 Schoenberg's Grids

Weber's chart of tone centers anticipates the schematic view of tonal regions given in Arnold Schoenberg's *Structural Functions of Harmony* (1948). In the first instance, Schoenberg labels his regions according to their functional harmonic names (tonic, dominant, subdominant, etc.), making the key relations generic rather than specific (see Figures 5.4 and 5.5.). Schoenberg redrafted his own charts, substituting specific key names. He offers separate charts for major and minor tonalities.

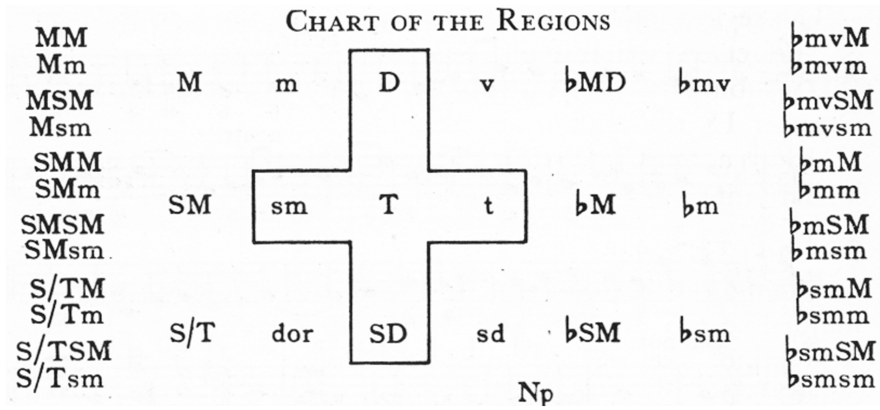

Figure 5.4. *Arnold Schoenberg's generic chart of tonal regions in the major mode (1948) is based on functional harmonic labels (D = dominant, T = tonic, SD = subdominant, etc.). Triads which are minor are indicated in lower-case letters.*

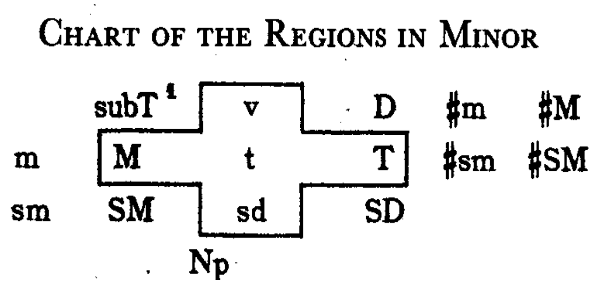

Figure 5.5. *Schoenberg's generic chart of tonal regions in the minor mode (1948). The schematic view is much simpler than that for major.*

Schoenberg's graphical view of the relationship of all relevant types of triads to a single tonal center implies degrees of distance from a tonic core. For Schoenberg, harmonic distance is computed in the first instance by the number of tones held in common by two scales. In the major mode, the dominant and subdominant are closely related to the tonic because (like the mediant and tonic minor) each holds six tones in common with it. The same proximities exist in the minor mode, but the chord qualities (major or minor) are reversed on the first six scale degrees.

5.1.4 Networks (*Tonnetze*) of Thirds and Fifths

The *Tonnetz*, a lattice structure first conceived by the mathematician Leonhard Euler (1766), has given rise to many spatial configurations of tonal relations. As the structure was elaborated in the nineteenth century through the work of such figures as Oettingen (1866) and Riemann (1877), it focused on major thirds and perfect fifths, either of which can be seen to provide a basis for circulating in tonal relations even though they also underpin important different results in the tuning of musical instruments.

Many selected sets of tones (e.g., those required by the pentatonic, diatonic, and chromatic scales), varying by the number of items and the intervallic distances between them, can be defined within subspaces of the *Tonnetz*. Conversely, supersets of pitch-classes, such as all those occurring in a given work, could be traced in the same theoretical spatial structure. The *Tonnetz* strengthens the difference between major and minor thirds by providing them with separate positions in and propagation pathways through its lattice. [For examples, see Ch. 6.]

The *Tonnetz* plays a role in many recent theories of tonal space, including those of Lewin (1987), Gollin (1998), Chew (2000), and Lerdahl (2001). The objectives of Lewin's theories are discussed extensively elsewhere. Gollin (1998), in order to explore tetrachord classes, has extended the Tonnetz into three-dimensional space. While our models of three-dimensional space adapt surface topographies of two-dimensional models on the surfaces of three-dimensional space, his are indigenously three-dimensional.

Chew's three-dimensional model (2000), which is derived from the *Tonnetz*, is the most intensely geometrical. Tones are lined up on a helix along the Circle of Fifths. The helix is circular in the x–y plane. It rises in the z direction. For a triad composed of three tones, she constructs the triangle whose vertices are provided by the constituent tones of the triad. Then the triad is represented by the weighted center of gravity of the triangle. In the same way, a key is represented as the center of gravity of the triangle whose vertices are the points identified with the three main triads (tonic, dominant, and subdominant) of the key. Finally, we observe three nested spirals—of tones, triads, and (major) keys—escalating in fifths. The Circle of Fifths, curling around the tube in Figure 5.8, can be identified with the innermost spiral of major or minor keys in Chew's model. [For Chew's own explanation, see Ch. 4.]

For Lerdahl (2001), five nested circles in a plane represent a hierarchy of tone sets. From the outermost to the innermost circle the constituent entities are (1) the chromatic scale tones, (2) the diatonic scale tones, (3) the basic triad, (4) the basic fifth, and (5) the keynote identified with the center of the circles. Based on this representation, Lerdahl arrives at the concept of the basic space of a chord by accumulating its fundamental, basic fifth, chord notes, underlying diatonic scale, and chromatic scale. Lerdahl calculates distances between key regions as distances between respective chords. He derives his measure from the combination of two criteria: (1) distance on the Circle of Fifths and (2) number of common tones. He considers the distance of the fundamentals separately on two circles of fifths. One of these represents the chromatic scale, the other the underlying diatonic scale. Instead of merely calculating the common chord notes, Lerdahl subtracts their basic spaces.

5.1.5 Key Relationships: A Summary

The choice of a geometrical space, the configuration of its shape, and the complexity which inheres in it necessarily imply closer connections between some sets of keys than others. Does the choice of geometries dictate and/or limit possibilities for implied proximity? Or does the theory inherently dictate the choice of a shape? Every theorist proposes different criteria for evaluating key relationships. Those we mention here are those we consider to have been the most original in their choices of geometries.

Heinichen's approach concentrates on properties of the tonic triad. The resulting criterion for key proximity is the number of common tones shared by tonic triads. The three non-identical pairs of major/minor triads sharing two common tones can be identified with the keys of the relative major/minor, the parallel major/minor, and the key of the mediant.[4] However, Heinichen's Circle of Fifths can be linked to the *Tonnetz* by extending the horizontal axis of fifths [in just intonation], replacing each tone by the corresponding major key and its relative minor key, and equating $g\sharp$ and $a\flat$.

Weber emphasizes the number of tones held in common by two scales. In the harmonic-minor scale, according to his measure, the keys of the dominant, subdominant, and relative major are considered to be close to the tonic. Proximity is judged by the number of shared pitch-classes, which brings parallel major and minor keys into a close relationship, for he emphasizes that both keys share the same principal scale degrees (I, IV, and V).

Schoenberg (1948) places keys into five classes based on their distance from the tonic. The classes are defined as (1) direct and close, (2) indirect but close, (3) indirect, (4) indirect and remote, and (5) distant. Class 1 requires a minimum of five common tones between the respective scales. Class 2 requires a minimum of three tones in common with the scale of the tonic. Criteria for the other three classes are multiple and dependent primarily on the complexity of modulatory pathways.

5.2 Toroidal Models from Theory to Application

5.2.1 Derivation of a Torus from Tonal Theory

A toroidal model of key relations is implicitly contained in Weber's charts of tone centers. To explain why the surface of a torus supplies a suitable composite model for major and minor keys, we reconfigure Weber's chart (Fig. 5.3) in three stages. The strip in Figure 5.6 is cut out from Weber's chart, then rotated from a vertical to a horizontal orientation.

In this schema, some keys and their enharmonic equivalents appear in multiple places. For example, G Minor appears in the upper row (Cell 5) in Figure 5.7 as well as in Cell 8 in the lowest row. In order to arrange a *unique* position for each key, we have to apply a spatial transformation. We do this by curling up the strip, forming a tube in order to unite the minor keys of the upper row with those of the lower one. When the two-dimensional strip in Figure 5.6 is curled as in Figure 5.7, single tonalities (such as G Minor in Curls 1 and 2 and B Minor in Curls 2 and 3) become redundant. This enables the formation of a tube, as shown in Figure 5.8.

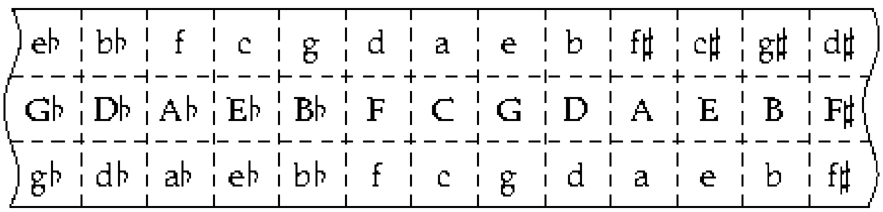

Figure 5.6. *Strip of keys cut out from Weber's chart of tone centers, and rotated to horizontal orientation. Compare with Figure 5.3.*

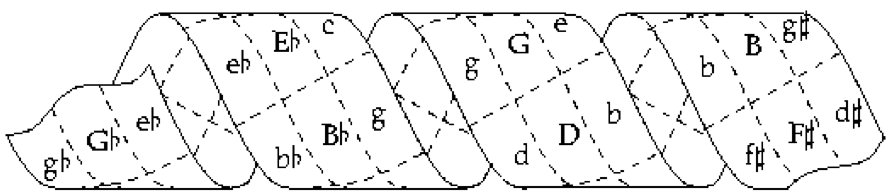

Figure 5.7. *G Minor occurs as the relative minor of B ♭ Major (most prominent row of Curl 1) and as the parallel minor of G Major (top row of Curl 2). When the strip is curled as above, these can be overlaid. Since the phenomenon is general, the entire rightmost column of Curl 1 can subsume the leftmost column of Curl 2, and likewise the corresponding columns of Curl 2 and Curl 3, to produce a tube, as shown in Figure 5.8.*

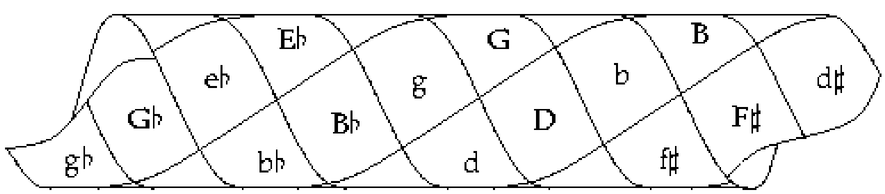

Figure 5.8. *Once the redundant positions are eliminated by compacting the curls into a single tube, the enharmonic equivalents at both ends of the tube can be wrapped horizontally to produce a torus.*

Since each of the original strips is a continuum in which (as in the Circle of Fifths) the original tonality eventually recurs, the duplicated keys at both ends of the tube may be joined to form a donut (torus). In this three-dimensional model of key relations (which we call a ToMIR, which stands for a topographically-ordered model of key relations), fifth-relations and third-relations are both preserved.

5.2.2 Topology of a Toroidal Surface

A toroidal surface can be parameterized in different ways. The most prominent are the following:

- A four-dimensional representation: In this form the toroidal key arrangement is first established in Krumhansl and Kessler (1982). Since a four-dimensional space is hard to imagine, Krumhansl and Kessler use their finding about the structure to further scale down the data to a two-dimensional representation (see Fig. 5.6).

- A three-dimensional representation (Fig. 5.9). This is the geometrical object that one would usually think of when talking about a torus. For the sake of a homogeneous configuration, the two- or the four-dimensional representation should be used; see the mathematical remark below and Krumhansl and Kessler (1982: 345).

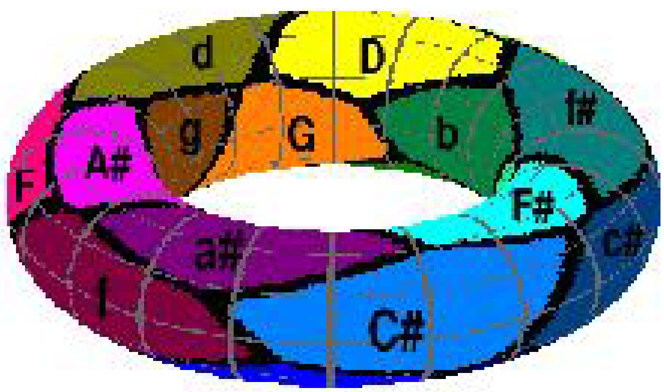

Figure 5.9. *A three-dimensional representation of key relationships constructed by connecting the upper and lower sides and the left and right sides.*

- A two-dimensional representation (cf. Figure 5.10). Each point of the toroidal surface is uniquely determined by two angles in Figures 5.9 and 5.10. So another parameterization is given by the set $[0, 2\pi r_1] \times [0, 2\pi r_2]$ endowed with the toroidal metric.

MATHEMATICAL REMARK: The two- and three-dimensional representations are isomorphic to each other. They are not isomorphic to the four-dimensional version, but the induced topological spaces are homeomorphic.

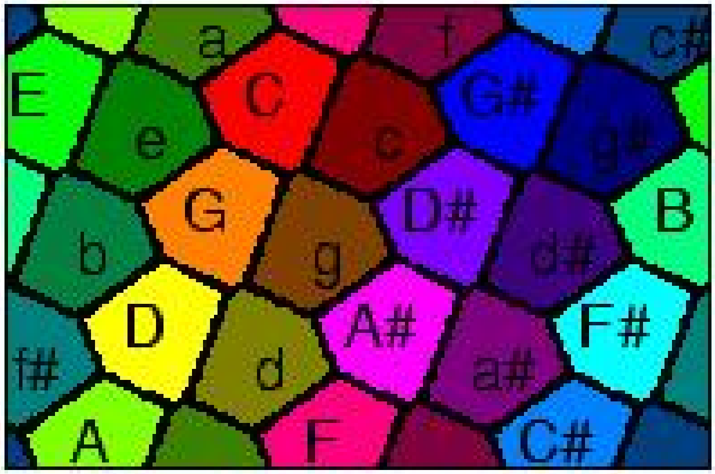

Figure 5.10. *A two-dimensional representation of key relationships.*

5.3 Perceptual Models of Pitch Relationships

5.3.1 Pitch-Class Usage Profiles

How are tonal relations understood by perceiving subjects? Much recent literature and current research addresses this subject from a psychological perspective. Much of this work is conducted not at the level of articulation afforded by the systems of earlier centuries, in which modes and keys are carefully distinguished, but instead at the more general level of pitch-classes divorced from a necessary grounding in any particular key.

By reducing enharmonic and octave equivalences to the values of (equally-tempered) chromatic tones within one octave, one can easily yield the twelve pitch-classes which correspond, in modern usage, to MIDI key numbers. In a 12-bin array, the notes of a C-Major scale would be represented by a 1 in the 1st (*c*), 3rd (*d*), 5th (*e*), 6th (*f*), 8th (*g*), 10th (*a*), and 12th (*b*) components, while a 0 is given for all other pitch-classes. Such twelve-dimensional binary vectors can be considered pitch-class sets.

In distinction to the set theory of Forte (1973), we seek to give usage weights for each pitch-class within the set. We know from widely reported histograms of the usage of tones within keys that in actual usage the tonic (Element 1) is most prominent, followed by the dominant (Element 8), then by the major third (Element 5).

In all tonal music, the first degree is the most prevalent. In the major mode, the second most prevalent scale degree is the fifth, while in the minor mode it is the minor third. Non-diatonic notes have repeatedly been shown in incidence studies to be relatively unimportant. Pitch-class preferences vary to some extent according to repertory and other musical factors. Histograms of pitch usage may be correlated with modal and scalar features (e.g., major or minor modes; whole-tone or other non-diatonic scales). [See ¤zmirli's study of pitch-class in relation to timbre, Ch. 1.]

5.3.2 Probe-Tone Weightings

By reducing pitch information to the base-12 level (cf. section 5.3.1 above), we can explore possible concordances with psycho-acoustical research. Probe-tone ratings (Krumhansl and Shepard 1979; Krumhansl and Kessler 1982) give a quantitative description of key that offers the possibility of relating statistical or computational analyses of music to cognitive psychology. Krumhansl (1990: 66–76) observes that each component in the probe-tone rating vector corresponds to the frequency of occurrence *and the cumulative duration of occurrence* of the corresponding pitch-class at *metrically prominent positions* in a tonal piece.

Lerdahl's basic space resonates with Krumhansl's probe-tone ratings (1990) and to pitch-class profiles. To compare two chords, the difference between the corresponding basic spaces is calculated. This is effectively the same as correlating the corresponding pitch-class profiles. Key distances are calculated by comparing the corresponding probe-tone ratings by correlation, Euclidean distance, or other dissimilarities described in Purwins (2005). Our Euclidean scaling is shown in four dimensions in Figure 5.11.

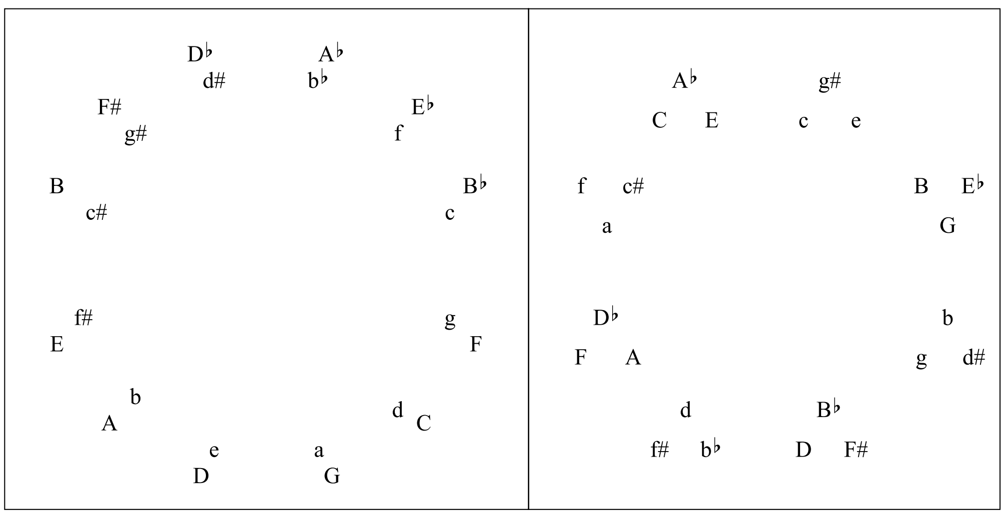

Figure 5.11. *Probe-tone ratings are reduced to a four-dimensional Euclidean space by multidimensional scaling. The scaled points lie approximately on a sub-manifold formed by the cross product of two cycles (left graph: Dimensions 1 and 2; right graph: Dimensions 3 and 4).*

Key kinship based on the number of common pitch-classes appears in the graphical representation of results from Krumhansl's probe-tone experiment (1990) in Figure 5.12. Lerdahl (2001) combines arguments of this section to derive a measure for key region similarity.

We perceive the harmonic graphing of probe-tones to incorporate both sets of major-minor relations together with tonic-dominant-subdominant relations.

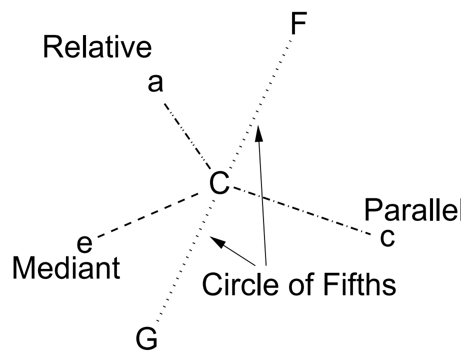

Figure 5.12. *The implicit two-dimensionality in the key visualization of Krumhansl and Kessler (1982). One linear axis corresponds to the Circle of Fifths, the other to the heterogeneous axes of parallel and relative third major/minor relationships. Here the key of the mediant (E Minor, relative to C Major) is considered to be adjacent to the tonic.*

The pitch-class material selected has a great impact on the style and character of a musical piece, e.g., the pitch-class material of the major, minor, whole-tone, or blues scales. Despite the complex inter-relationships between major/minor tonality and such musical features as voice leading, form, or beat strength, the frequency of occurrence of individual pitch-classes is a major cue for the percept of a tonality (Krumhansl 1990).

5.4 Pitch Incidence and Audio Key-Finding

Recent audio studies by Purwins et al. (2004a) involving a reduction of audio signals to a 12-tone mapping give provisional results for the selective use of particular pitch-classes by certain composers. One aim is to arrive at an algorithm for automatic key-finding in audio data. We developed the concept of the constant-quotient (CQ) profile (Purwins et al. 2000b), which is a 12-tone vector similar to a probe-tone rating scale. A CQ calculation can be made quickly, has been shown to be stable over a wide range of recordings, and the profiles created are transposable. We have developed short- and long-term profiles and have created a CQ reference set. On the basis of this work, we have trained self-organizing maps (SOMs) on the reference values, then used these for audio-tone-classification purposes.

To test our approach, we investigated the type and amount of information captured in CQ profiles from complete but short pieces of piano music (Purwins et al. 2004b). The analyzed data set favors cycles of works which are ostensibly evenhanded in their distribution of keys. It includes pieces from Bach's *Well-Tempered Clavier* (WTC), Books I (1722) and II (*c*. 1740), Chopin's *Préludes* Op.28 (which also consist of one piece in every key, 1838–39), Alkan's *Préludes* Op.31 (exploring all major and minor keys, 1847), Scriabin's 24 *Preludes* Op.11 (1888–96), Shostakovich's Op.34 (1932–33), and the fugues of Hindemith's *Ludus tonalis* (consisting of one fugue for each pitch-class, purposely avoiding a clear use of major or minor; 1942). We employ supervised and unsupervised machine-learning techniques.

To examine divergent performance profiles, we have evaluated different performances of the same works. Our database for the audio work currently contains 226 CQ profiles. For the *Well-Tempered Clavier*, we have profiled the performances of Samuil Feinberg and Glenn Gould. Chopin's *Préludes* Op.28 were profiled from performances by Alfred Cortot and Ivo Pogorelich. Profiles of Alkan's 25 *Préludes* Op.31 are derived from performances by Olli Mustonen. Scriabin's *Preludes* Op.11 have profiles obtained from performances by Scriabin himself on a Welty-Mignon piano disk as well as from three other pianists. The twelve fugues in Hindemith's *Ludus Tonalis* are performed by Mustonen.

We have mapped tonal data to both scale degrees and to pitch-classes, according to the nature of the question posed. The supervised approach attempts to identify composers from automatically classified profiles. In the case of Bach, we find that all pitch-classes are significant, while in Chopin's piano music there is a heavy reliance on the third, sixth, and seventh of the twelve pitch-classes. For Shostakovich, the third pitch-class is singularly important. Scale-degree profiles for Bach, Chopin, and Hindemith are shown in Figure 5.13.

Unsupervised methods provide (1) a cluster analysis, leading to one major and one minor cluster, and (2) a visualization technique, Isomap, which reveals in its two-dimensional representation some additional harmonic structure apart from major-minor dualities. Overall, we are astonished by the amount of information found in the profiles. All of this is retrievable in a straightforward manner from any digital recording. It is important to indicate the performer, since the cumulative duration of individual pitches varies by performer.

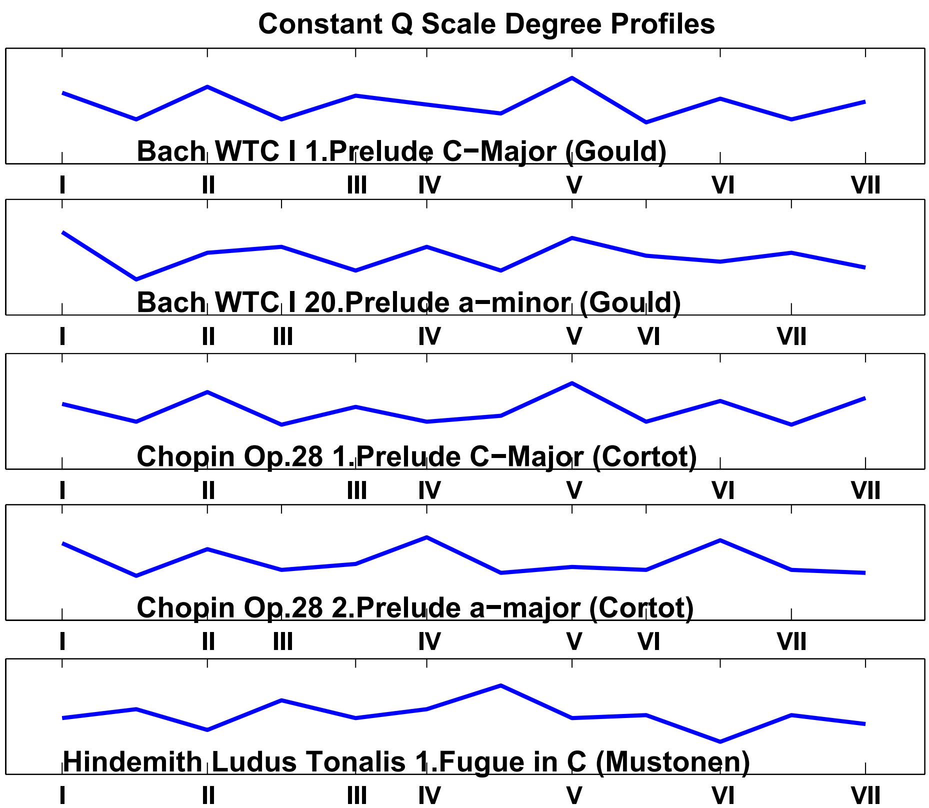

Figure 5.13. *Profiles by scale degree. Transposed constant-quotient (CQ) profiles for selected pieces from Bach, Chopin, and Hindemith. Scale degrees are shown on the horizontal axis. In Chopin and Bach the peaks are related to the diatonic scale and to probe-tone ratings. Hindemith de-emphasizes the diatonic notes.*

On a general level, pitch-class usage in tonal repertories can be skewed by composer preferences for particular keys. This is evident from ordinary bibliographical information. We can see significant differences in the keys used by such composers as Vivaldi, Bach, and Chopin (Figure 5.14). Part of the effect we see in this figure comes from differences of instrumental medium—a predominantly string ensemble in Vivaldi, a mixture of harpsichord and organ in Bach, and predominantly modern piano in Chopin. Such associations may reflect the influences of timbre and mechanical convenience on pitch choice.

In Figures 5.15 and 5.16 we show this variability in the preludes in major and minor keys in Book I of the *Well-Tempered Clavier*. In dealing with audio data, it is also important to compute total durations for pitch-classes (and/or scale degrees) to accommodate performer-specific variations in treating tempo and duration.

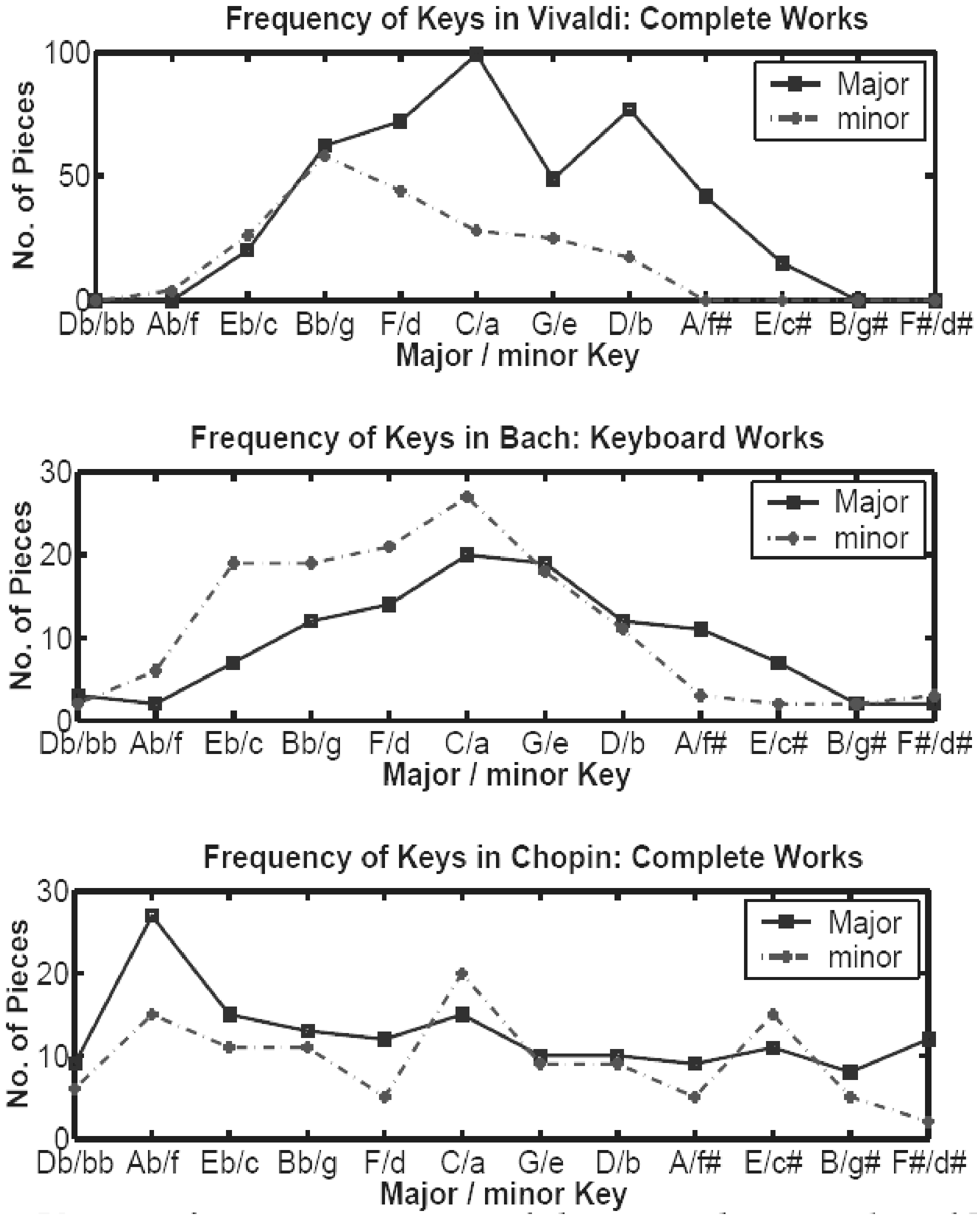

Figure 5.14. *Profiles by relative major/minor groupings for key preferences in works by Vivaldi, Bach, and Chopin. In these graphs relative major/minor pairs are given a single value.*

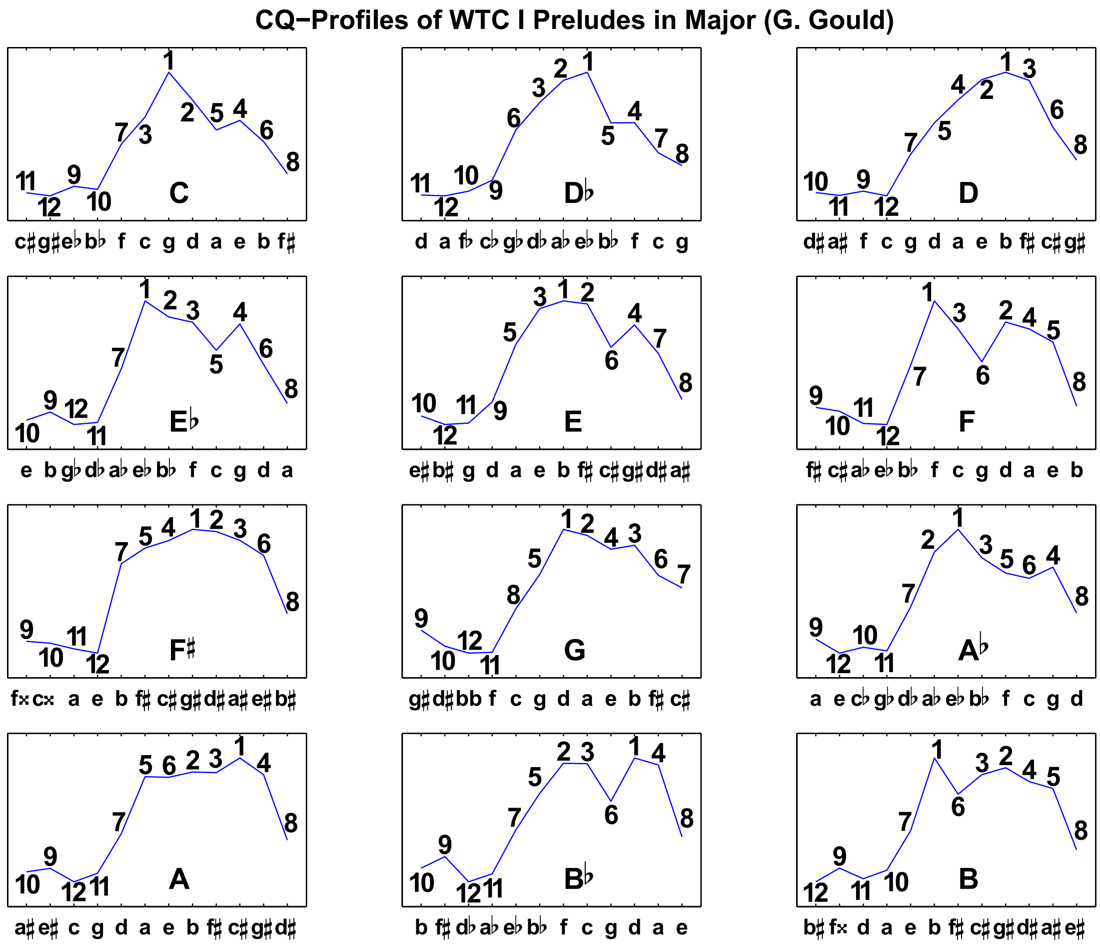

Figure 5.15. *Constant-quotient profiles ordered by pitch-class for major-key preludes (as performed by Glenn Gould) from Book I of Bach's Well-Tempered Clavier. It can be observed that most profiles can be generated from the combination of a big peak centered on the first scale degree and a small peak on the (major) third degree.*

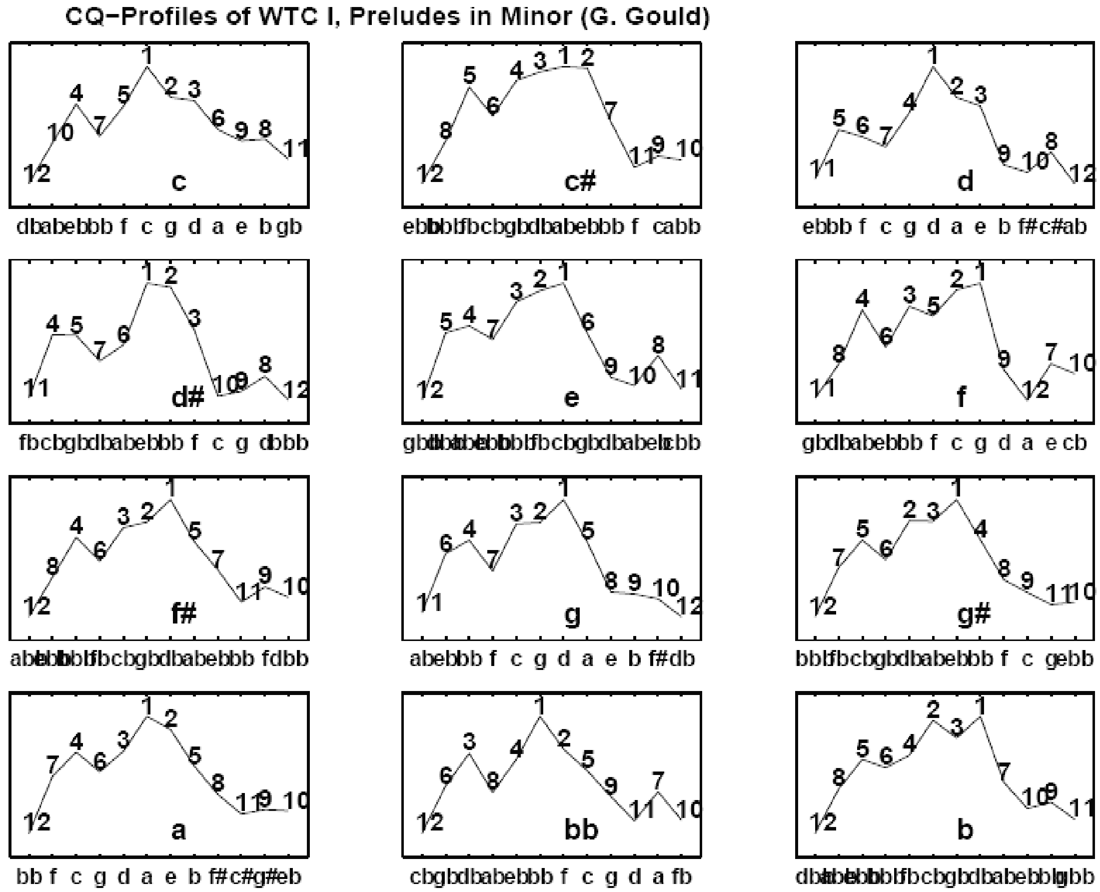

Figure 5.16. *Constant-quotient profiles ordered by pitch class for minor-key preludes (as performed by Glenn Gould) from Book I of Bach's* Well-Tempered Clavier.

If we look at the fugues of Book I of the *Well-Tempered Clavier*, we find that the Circle of Fifths which emerges from our profiles is not so regular as in drawings by theorists. In terms of sounding time (each pitch-class multiplied by the accumulation of all its durations), the geometrical correspondence is also not so close as one would anticipate (Figure 5.17).

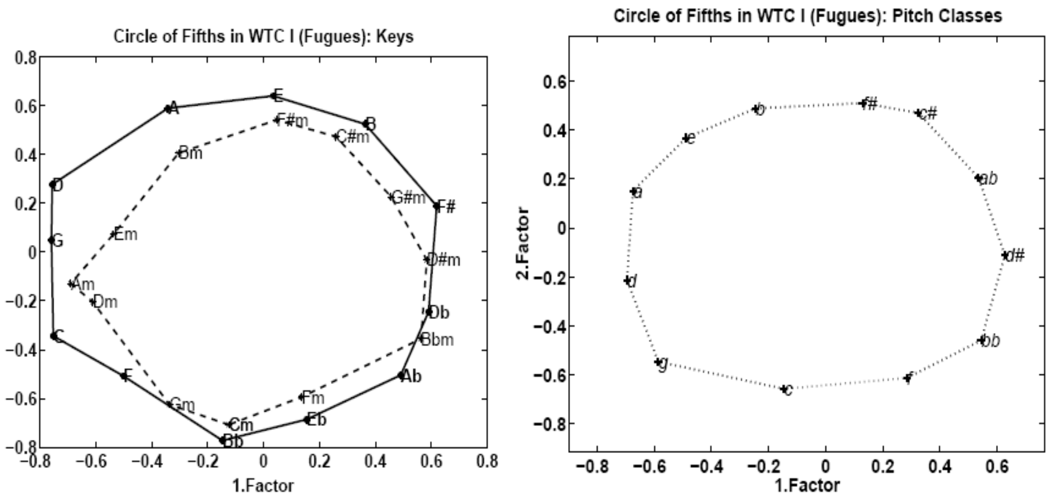

Figure 5.17. *Distributions by key and pitch-class compared. Individual plots for the distribution and elapsed duration of (a) keys and (b) pitch-classes in fugues of Book I of Bach's* Well-Tempered Clavier.

5.5 Other Applications of Toroidal Models

We have explored many areas of application for the approach described above. Here we mention briefly several procedures used in the visualization of complex relationships and their applicability to the study of key and pitch relationships in tonal music. Many more will be found in our recent publications.

5.5.1 Self-Organizing Maps

By means of the self-organizing feature map, a simple binary notion of close and distant keys induces geometric topologies consistent with psychological experiments and writings in music theory. In addition to the assumptions made by Krumhansl and Weber, we can also model other notions of proximity. We could, for example, assign different real-distance values for the keys of the dominant or of the parallel and relative minor/major. In comparison to Lerdahl (2001), the model presented here is non-hierarchical and much simpler.

Our learning algorithm (Blankertz et al. 1999b) uses Kohonen's idea of establishing a topology-preserving map in a new and unusual manner. In the usual self-organizing map (SOM; Kohonen 1982), the objects under investigation are characterized by feature vectors in a high-dimensional space. The vectors are assumed to lie approximately on a low-dimensional sub-manifold outlined by the given metric on the formal neurons (after successful training). The correspondence between neurons and objects is established by the self-organizing learning process.

In our modified SOM, each neuron represents exactly one object in a fixed *a priori* correspondence. In the learning process, a suitable placement of the neuron vectors is to be found in a stipulated metrical space that realizes a given neighborhood relation on the neurons[5]. The particular assumptions about proximity made by our application are coded in the function and affect neighboring functions, which results in the updating of weights and the establishment of a "winning neuron." It is handled separately in order to approximate the shape of a triangle.

5.5.2 Neuroscientific Correlates of Key Relations

Both the SOM and the topographic ordering map can be used to map a set of formal neurons to a set of input vectors by selecting an input vector with minimum Euclidean distance to the given neuron vector. Usually, an SOM displays a direct visualization of this mapping.

To get a smoother display, we extend this mapping to the whole manifold that is outlined by the neurons by means of interpolation. That is, we associate with each input vector a possibly unconnected region on the manifold. In simulations, that manifold is a toroidal surface and there are always 24 input vectors, each representing a major or minor key. We use 21×12 formal neurons supplied with a toroidal metric. Neurons are arranged in a mesh, gluing together opposite borders. A point is colored black if the distance (of its projection) to the nearest input vector is not sufficiently smaller than the distance to the second nearest. So each region gets a black border whose thickness corresponds to uncertainty, in a relative measure. The placement of the key name is determined by the position of the neuron with minimum distance to the input vector that corresponds to that key.

In an fMRI study, using the representation of a tone center parameterized by two toroidal angles on the ToMIR, Janata et al. (2002) find evidence for localizing brain activity related to tone-center modulation. Textbook-like modulations, synthesized with FM-clarinets, are played to the subjects. Two toroidal angles on the ToMIR seem to represent tone centers preserving sufficient information to identify voxels that are sensitive to tone-center transitions. Currently, non-invasive brain-imaging techniques do not yet seem able to indicate whether voxels that are sensitive to distinct tone centers are spatially arranged in the brain in a certain manner (e.g., in a torus) or in any other (e.g., more dynamic) configuration. An actual neural correlate of a dynamic topography may not give equal space to all tone centers, since tonic and dominant will occur more often than remote key regions. Also, modulation from one scale degree to another one is not usually matched by the reverse transition. That is, the harmonic progression I–vi occurs more often than that of vi–I.

Is there a neurobiological correlate of tone-center processing? Most auditory models cover only roughly the first stages of auditory processing. The topographic principle in the SOM (Kohonen 1982) corresponds to tonotopy (Schreiner and Langner 1988) in the auditory domain. Because it is considered as a model of the cortex, it is an extreme simplification, simulating only topography. We think more knowledge about neural processing of sound is needed to make a well-grounded hypothesis on the exact representation of tone centers in the cortex.

5.5.3 Correspondence Analysis

In our efforts to show the emergence of the Circle of Fifths and of a ToMIR from CQ profiles extracted from Bach's *Well-Tempered Clavier* and Chopin's *Préludes* (Purwins et al. 2000b, 2004b, 2005), we have found that correspondence analysis and Isomap are useful substitutes for an SOM because they serve as metaphors for cortical organization.

We became involved with correspondence analysis in order to embed distinct musical entities such as tones, triads, and keys in a common plane or space, in order to make their relationships graphically obvious. In Purwins et al. (2004b), correspondence analysis was used to investigate the mutual relationships between keys and pitch-classes. The embedding of keys and pitch-classes can then be displayed in biplots. With a dissimilarity at hand, keys represented by their probe-tone ratings can be visualized. We give an example from Book I of the *Well-Tempered Clavier* in Figure 5.18.

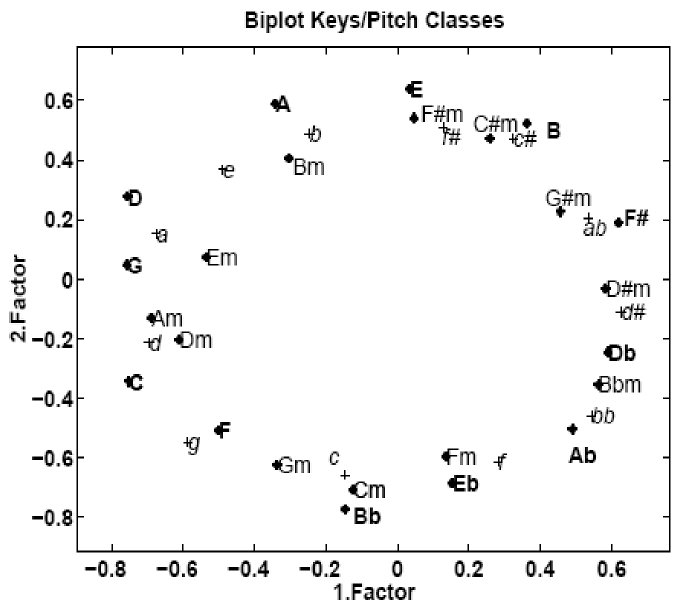

Figure 5.18. *Overlaid plots of key and pitch-class usage in the fugues of Bach's* Well Tempered Clavier*, Book I. Cf. Figure 5.17.*

5.5.4 Toroidal Simulations

We simulate the toroidal configuration from tone centers considered to be close to each other according to Weber. Those are the dominant, subdominant, relative, and parallel kinships. For the topographic ordering map, the appropriate set of close relations is

$$V_1 := \{ \text{C} - \text{G}, \text{C - F}, \text{C - a}, \text{C - c}, \text{c - g}, \text{c - f}, \text{c - Eb}, \ldots \}$$

where the dots stand for analogous relations between keys with different tonic key-notes. Even in this setting, where the set V_1 is the only information given about the structure of inter-key relations, the algorithm ends up with the arrangement depicted in Figure 5.19. This simulation resembles a global arrangement of tone centers, like that which evolves in Weber's chart, but is not predetermined.

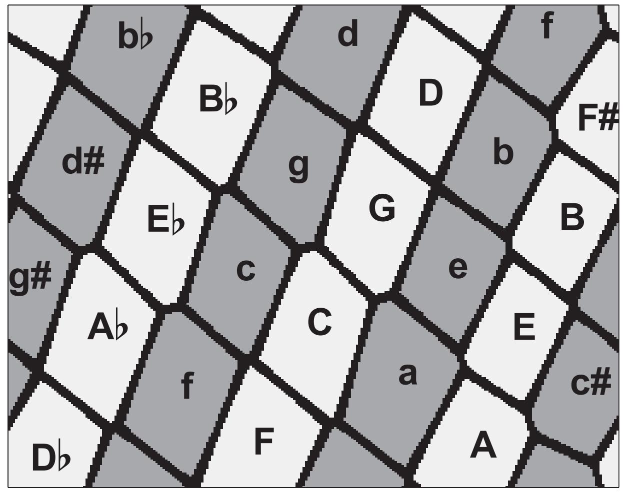

Figure 5.19. *Arrangement of keys evolving from sets of close relationships by the topographic ordering map. Relationship* V_1 *resembles the local structure of Weber's chart of tone centers (1817). Compare previous figure.*

For comparison, let us consider relations between tone centers resulting from maximization of common tones between adjacent scales as well as maximizing common tones between their respective tonic triads. The tonic's kin, the dominant, subdominant, relative, and parallel, are now additionally joined by the mediant. Under this stipulation, E Minor is as well an immediate neighbor of C Major, as are F Major, G Major, and A Minor. To integrate the strengthening of the mediant relation, we expand the set of close relationships to

$$V_2 := V_1 \ ? \ \{C - e, c - G\#, \ldots\}$$

This is consistent with Krumhansl's probe-tone ratings (1990: 39, 46) and shows a strong correlation between the tonic and its mediant (cf. Figure 5.12). Of course, the small amount of information that is used in this simulation is not sufficient to produce finer distinctions. In Krumhansl's arrangement, A Minor, for example, is closer to C Major than E Minor.

In our color illustrations, Scriabin's color-mapping has been used to good effect, for example, in the depiction of Chopin's *Préludes* Op.28 as performed by Alfred Cortot (1932–33).

5.5.5 Studies in Music Theory and Perception

The curling of Weber's chart to form a torus can be subtly related to tone-center modulation paths and key symbols. For example, the key architecture of Wagner's *Parsifal* represents a spiritual and physical journey from A♭ Major to A♭ Major (Lerdahl 2001: 119 ff; Purwins 2005). The path from Earth to Heaven (vertical axis) is by rising perfect fifths/descending major fourths (vertical axis), while from Evil to Good (horizontal axis) it is by rising major sixths/ descending minor thirds (horizontal axis). See Figure 5.20.

Heaven

	D	b	B	g♯	A♭	f	F	d	D	
	G	e	E	c♯	D♭	b♭	B♭	g	G	
	C	a	A	f♯	F♯	e♭	E♭	c	C	
	F	d	D	b	B	g♯	A♭	f	F	
	B♭	g	G	e	E	c♯	D♭	b♭	B♭	
	E♭	c	C	a	A	f♯	F♯	e♭	E♭	
Evil	A♭	f	F	d	D	b	B	g♯	A♭	Good
	D♭	b♭	B♭	g	G	e	E	c♯	D♭	
	G♭	e♭	E♭	c	C	a	A	f♯	G♭	
	B	g♯	A♭	f	F	d	D	b	B	
	E	c♯	D♭	b♭	B♭	g	G	e	E	
	A	f♯	F♯	e♭	E♭	c	C	a	A	
	D	b	B	g♯	A♭	f	F	d	D	

Earth

Figure 5.20. *The intersection of the Good-Evil and Heaven-Earth axes (all initially in A♭ Major) at D Major in the overall harmonic plan of Richard Wagner's* Parsifal*, according to our method of analysis.*

Physically, A♭ Major has a unique location on the ToMIR. Conceptually, A♭ Major has different meanings associated with four separate positions on Weber's chart. What is most significant in Wagner's plan, when considered in the context of Weber's extended chart, *Parsifal* depicts in tonal space the cross of spiritual journey, with a convergence of both paths at D Major (the tritone of A♭). The tritone interval signifies the wound of Amfortas.

Representations of key regions can be used to track paths of modulations, tonicizations, and applied dominants within a piece (cf. Purwins et al. 2000b, Purwins 2005, Toiviainen 2005, Lerdahl 2001, Cohn 2007). Toiviainen (2005) shows images of instantaneous tonal activation. In addition, he uses self-similarity based on SOM activations for visualizing the tonal structure of the piece [cf. Ch. 10].

5.6 Discussion and Conclusions

The central result of this research is that the hypothesis of psycho-physical parallelism for the topographic ordering map of tonal harmonic (interkey) relations (TOMIR) is supported in a generalized form, maintaining a comparable level of biological relevance. When our operational model for the acquisition of a mental sense of tonal harmonic relations is exposed to pieces of tonal music, a Circle of Fifths and a homogeneous toroidal configuration of keys evolve. The validity of our hypothesis is demonstrated on a broad variety of samples (actual recordings of performed music by Bach and Chopin) as opposed to a limited number of cadential patterns.

The model's basic assumptions are reduced to (1) logarithmic frequency resolution, (2) contextual formation of the semantics of tone-centers (through the application of the Gestalt principle of proximity in algorithms such as Isomap and correspondence analysis), (3) identification of octave components based on the projection of pitch onto the pitch-class circle of the helical model, and (4) the consideration of semitones. Overall, fewer assumptions are needed to verify our hypothesis than in previous approaches: the toroidal structure is not stipulated in the architecture. The neuromimetic model and the CQ-model are of comparable neuromimetic relevance.

Surprisingly, it appears that the specific details of the auditory process, as implemented in an auditory model, are not particularly relevant for the cognitive development of a sense of tonal harmonic relations. Although one of the main functionalities of Meddis' hair-cell model (1988) is the response to onset, sustain phase, and offset, these features appear to be of minor relevance for the establishment of a sense of overall tonal content. Since for autocorrelation the biological plausibility is questioned, we use CQ-profiles, without loosing confirmed biological relevance. CQ-profiles are consistent with the Weber-Fechner rule of logarithmic perception of frequency and they reveal high coherence with probe tone ratings in music psychology (Purwins et al. 2000b).

Toroids offer a powerful model for the analysis of key relationships in tonal music and in its performance. The toroidal model is particularly useful in its ability to subsume both the kinds of tonal relationships proposed by music theorists, based on intervals and their combinations in chords, and by psychologists in recent experimental literature on music perception, where it is based largely on pitch-class ratings.

In the latter connection, we find that audio mapping of data of high-dimensionality, when reduced to one dimension, can provide an adequate basis for practical discriminations between performances. Many areas of application can benefit from the coordination of data representations bearing on the construction of this spectrum of more specific and more general definitions of pitches and keys. All circular models imply through their design particular interpretations of key proximity. The details of relative relationships will vary according to the particular graphical configuration of the constituent items.

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Notes

1. The use of the helix to explain pitch relations can be traced back at least as far as Drobisch (1855).

2. For details on the early history of tonal music theory, see Werts (1983). Werts' analysis of modulations in a large corpus of music also evaluates modulatory goals and finds that modulations to dominant, subdominant, parallel, and relative keys are the most common.

3. On Weber's theoretical framework, see Saslaw (1992), who in recent writings has linked his geometrical approach to concepts in cognitive musicology.

4. We do not differentiate between a key and a tone center, but note the variations in the usage of Schoenberg (1948) and Lerdahl (2001), who refer to key regions or key areas. In Lerdahl's terminology, we reduce key to Level D of the basic space (Lerdahl 2001: 47, Figure 2.4).

5. Our algorithm generalizes to the case of arbitrary neighborhood degrees $dK(i,j) \in R > 0$ in a straightforward manner for real numbers R.

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